Correction Model Midterm Exam
Linear Algebra 2 05/03/2018

1. (a) An inner product on V is a function from $V \times V$ to $\mathbb{R}$ assigning to each $(v,w) \in V \times V$ a real number $\langle v, w \rangle$ such that

② (i) for all $v \in V$ $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$,

① (ii) for all $v, w \in V$ $\langle v, w \rangle = \langle w, v \rangle$,

② (iii) for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$
$$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$$

(b) Let $v, w \in V$. Let $\|v\| := \sqrt{\langle v, v \rangle}$ and

⑤ $\|w\| := \sqrt{\langle w, w \rangle}$. Cauchy-Schwarz states that
$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|.$$

(c) A norm on V is a function from V to $\mathbb{R}$ assigning to each $v \in V$ a real number $\|v\|$ such that

② (i) for all $v \in V$ $\|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = 0$.

① (ii) for all $v \in V$ and $\alpha \in \mathbb{R}$ $\|\alpha v\| = |\alpha| \cdot \|v\|$

② (iii) for all $v, w \in V$ $\|v + w\| \leq \|v\| + \|w\|$.

(d) In order to check that $\sqrt{\langle u, u \rangle}$ defines a norm on V we check the three properties:

① (i) Let $u \in V$. Then $\|u\| = \sqrt{\langle u, u \rangle} \geq 0$ and
$$\|u\| = 0 \Leftrightarrow \sqrt{\langle u, u \rangle} = 0 \Leftrightarrow \langle u, u \rangle = 0 \Leftrightarrow u = 0$$

(ii) Let $u \in V$ and $\alpha \in \mathbb{R}$. We have

2 $$\|\alpha u\| = \sqrt{\langle \alpha u, \alpha u\rangle} = \sqrt{\alpha^2 \langle u, u\rangle}$$
$$= \sqrt{\alpha^2} \cdot \sqrt{\langle u, u\rangle} = |\alpha| \cdot \|u\|$$

(iii) Let $u, v \in V$. We have

2 $$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle$$
$$+ \langle v,u\rangle + \langle v,v\rangle = \|u\|^2 + 2\langle u,v\rangle + \|v\|^2.$$

Now, by Cauchy-Schwarz, we have

$$\langle u, v\rangle \leq |\langle u,v\rangle| \leq \|u\| \cdot \|v\|.$$

Hence

$$\|u+v\|^2 \leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2$$
$$= (\|u\| + \|v\|)^2.$$

We conclude that

$$\|u+v\| \leq \|u\| + \|v\|.$$

2. $V = C[0,1]$, $\langle f, g\rangle = \int_0^1 f(x)\, g(x)\,dx$.

S subspace of V : $S := \{a + b\sqrt{x} \mid a, b \in \mathbb{R}\}$

a) A basis of S is $\{1, \sqrt{x}\}$. Apply GS to obtain an orthonormal basis $\{u_1(x), u_2(x)\}$.

$\|1\|^2 = \int_0^1 dx = 1$, so $\|1\| = 1$.

(2) Therefore, take $u_1(x) = 1$.

Next, project $\sqrt{x}$ onto span $(u_1(x))$ to get $p_1(x) = \langle\sqrt{x}, u_1(x)\rangle \cdot u_1(x)$ with

$$\langle\sqrt{x}, u_1(x)\rangle = \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}$$

(2) So $p_1(x) = \frac{2}{3}$.

(3) Now $\sqrt{x} - p_1(x) = \sqrt{x} - \frac{2}{3}$ is orthogonal to $u_1(x)$. Its norm squared is

$$\|\sqrt{x} - p_1(x)\|^2 = \int_0^1 \left(\sqrt{x} - \frac{2}{3}\right)^2 dx$$

$$= \int_0^1 x - \frac{4}{3}\sqrt{x} + \frac{4}{9}\,dx$$

$$= \left[\frac{1}{2}x^2 - \frac{2}{3}\cdot\frac{4}{3}x^{3/2} + \frac{4}{9}x\right]_0^1$$

$$= \frac{1}{2} - \frac{8}{9} + \frac{4}{9} = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$$

(3) $\Rightarrow \|\sqrt{x} - p_1(x)\| = \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$.

We then take $u_2(x) = \dfrac{\sqrt{x} - \frac{2}{3}}{\|\sqrt{x} - \frac{2}{3}\|} = 3\sqrt{2}\left(\sqrt{x} - \frac{2}{3}\right)$.

(b) This is the orthogonal projection of $y(x) = x$ onto the subspace S:

3 $$p(x) = \langle x, 1\rangle \cdot 1 + \langle x, 3\sqrt{2}(\sqrt{x} - \tfrac{2}{3})\rangle \cdot 3\sqrt{2}(\sqrt{x} - \tfrac{2}{3})$$

Now,

2 $$\langle x, 1\rangle = \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}$$

$$\langle x, 3\sqrt{2}(\sqrt{x} - \tfrac{2}{3})\rangle =$$

$$\int_0^1 3\sqrt{2}\left(x^{3/2} - \tfrac{2}{3}x\right)dx =$$

3 $$\left[3\sqrt{2}\,\tfrac{2}{5}x^{\frac{5}{2}} - 3\sqrt{2}\,\tfrac{1}{2}\cdot\tfrac{2}{3}x^2\right]_0^1 =$$

$$3\sqrt{2}\cdot\tfrac{2}{5} - 3\sqrt{2}\,\tfrac{1}{3} = 3\sqrt{2}\left(\tfrac{2}{5} - \tfrac{1}{3}\right) =$$

$$3\sqrt{2}\left(\tfrac{12}{30} - \tfrac{10}{30}\right) = 3\sqrt{2}\,\tfrac{1}{15} = \tfrac{1}{5}\sqrt{2}$$

Conclusion:

$$p(x) = \tfrac{1}{2} + \tfrac{1}{5}\sqrt{2}\cdot 3\sqrt{2}\left(\sqrt{x} - \tfrac{2}{3}\right)$$

$$= \tfrac{1}{2} + \tfrac{6}{5}\left(\sqrt{x} - \tfrac{2}{3}\right)$$

2 $$= \tfrac{1}{2} + \tfrac{6}{5}\sqrt{x} - \tfrac{4}{5} = -\tfrac{3}{10} + \tfrac{6}{5}\sqrt{x}$$

Of course, an answer is also correct if this is not computed "until the end"

3. a) Let $v \in V$. Then v can be written as a linear combination of the vectors $v_1, \dots v_k$:

$$v = x_1 v_1 + x_2 v_2 + \dots + x_k v_k$$

Define $x \in \mathbb{R}^k$ by $x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$. Then $v = Vx$

Thus $Av = AVx = VBx$. Denote $y := Bx$

Then $y \in \mathbb{R}^k$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}$.

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This yields $Av = Vy = y_1 v_1 + y_2 v_2 + \dots + y_k v_k$ which is a linear combination again of the vectors $\{v_1, v_2 \dots v_k\}$. We conclude $Av \in V$

b) Let λ an eigenvalue of B. Then there exists $y \neq 0$ such that $By = \lambda y$. Thus $VBy = \lambda Vy$ so $AVy = \lambda Vy$. If $Vy \neq 0$ then λ is an eigenvalue of A.

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Suppose $Vy = 0$. Write $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}$.

Then $y_1 v_1 + y_2 v_2 \dots + y_k v_k = 0$. Since $\{v_1 \dots v_k\}$ are linearly independent, we must have $y_1 = y_2 = \dots = y_k = 0$, so $y = 0$. Contradiction !! Hence $Vy \neq 0$ so λ eigenvalue of A.

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c) A nonsingular $\Rightarrow$ all eigenvalues of A nonzero $\Rightarrow$ all eigenvalues of B nonzero $\Rightarrow$ B nonsingular.

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Other solution: assume A nonsingular but B is singular. Then there exists a vector $y \neq 0$ s.t. $By = 0$. Hence $AVy = VBy = 0$. Now, $Vy \neq 0$ since the columns of V are linearly independent. So $AVy = 0$ with $Vy \neq 0$. This is a contradiction with A nonsingular.

d) We have then $Av_i = \lambda_i v_i$ for certain $\lambda_i \in \mathbb{C}$. This yields

$$AV = A(v_1 \cdots v_k) = (Av_1 \cdots Av_k)$$

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$$= (\lambda_1 v_1 \cdots \lambda_k v_k)$$

$$= (v_1 \cdots v_k) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$$

$$= V\Lambda$$

with $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$ diagonal. Hence $VB = V\Lambda$. This implies $B = \Lambda$ so B is diagonal matrix with the eigenvalues of A on the diagonal.

e) If $\{v_1 \dots v_k\}$ orthonormal set, then we must have $V^T V = I_{k \times k}$

5 Since $AV = VB$ we obtain $V^T A V = B$

Hence $B^T = (V^T A V)^T = V^T A^T V$

$= V^T A V = B$ so B is symmetric.

4. a) Let $x \in \mathbb{C}^n$ and denote $\alpha := x^H A x \in \mathbb{C}$

We then have

$$\bar{\alpha} = \overline{x^H A x} = (x^H A x)^H = x^H A^H x = -x^H A x$$
$$= -\alpha.$$

So we have that $\alpha \in \mathbb{C}$ satisfies $\bar{\alpha} = -\alpha$.

(5) Write $\alpha = a + bi$, with $a = Re(\alpha)$.

Then $a - bi = \bar{\alpha} = -\alpha = -a - bi$

This implies $a = -a$, so $a = 0$.

Hence $Re(x^H A x) = Re(\alpha) = a = 0$

b) Let λ be an eigenvalue of A, with eigenvector x, $x \neq 0$.

(5) We have $Ax = \lambda x$, so $x^H A x = \lambda \|x\|^2$

By (a) $Re(x^H A x) = 0$, so we have

$$Re(\lambda) = Re\left(\frac{x^H A x}{\|x\|^2}\right) = \frac{1}{\|x\|^2} Re(x^H A x)$$
$$= 0$$

c) By Schur's theorem there exists a unitary matrix U and an upper triangular T s.t.

$$A = U . T U^H$$

(5)

equivalently

$$U^H A U = T$$

We get $T^H = (U^H A U)^H = U^H A^H U = -U^H A U$
$$= -T$$

T is upper triangular so $T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & t_{nn} \end{pmatrix}$

We hence get

$$\begin{pmatrix} \bar t_{11} & 0 & \cdots & 0 \\ \bar t_{12} & \bar t_{22} & & \vdots \\ & & \ddots & 0 \\ \vdots & \vdots & & \\ \bar t_{1n} & \bar t_{2n} & \cdots & \bar t_{nn} \end{pmatrix} = T^H = -T = \begin{pmatrix} -t_{11} & -t_{12} & \cdots & -t_{1n} \\ 0 & -t_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & -t_{nn} \end{pmatrix}$$

This implies $t_{12}, t_{13}, \ldots, t_{1n} = 0$

$t_{23} \ldots t_{2n} = 0$

etc.

So T is diagonal.

d) U unitary means $U^H U = I$. Let λ be an eigenvalue of U. Then $Ux = \lambda x$ for some $x \in \mathbb{C}^n$, $x \neq 0$.

We then obtain

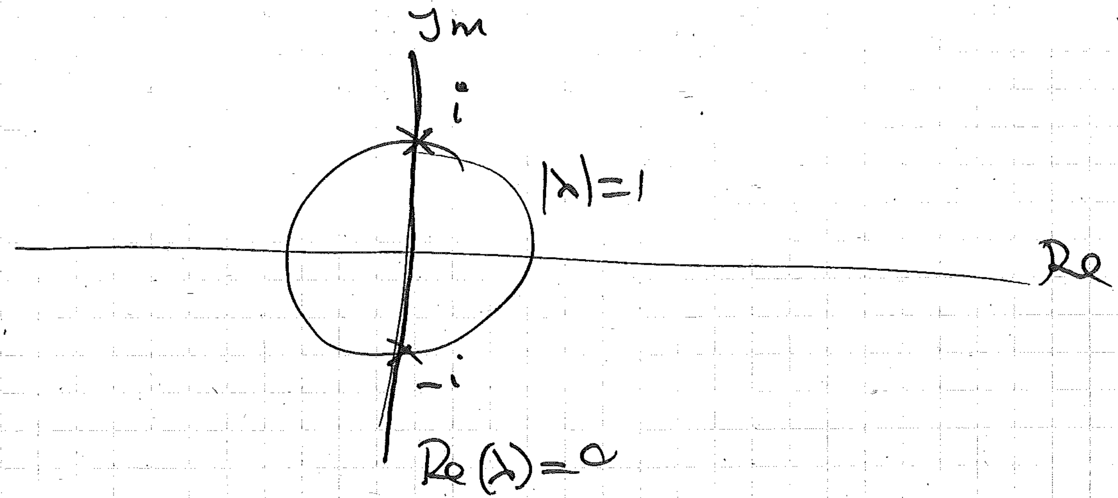
(5) $$(Ux)^H Ux = x^H U^H U x = x^H x = \|x\|^2.$$

on the other hand:

$$(Ux)^H Ux = (\lambda x)^H \lambda x = x^H \lambda^H \lambda x$$
$$= x^H |\lambda|^2 x$$
$$= |\lambda|^2 x^H x$$
$$= |\lambda|^2 \|x\|^2$$

Hence $|\lambda|^2 = 1$ so $|\lambda| = 1$

(5) e) $Re(\lambda) = 0$ and $|\lambda| = 1$ so either $\lambda = i$ or $\lambda = -i$



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She studied Applied Mathematics, master degree in 1994. Here! around Veldman, Verstappen, Wubs

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Natasha...